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Exact solutions of non-integrable lattice equations

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Abstract
The method of generalized conditional symmetry used by Fokas and Liu for deriving nonintegrable evolution equations which possess exact analytic solutions is here extended to the construction of differential-difference equations supporting two-kink and two-soliton solutions. In particular, we build the discrete analogue of a Burgers type equation with reaction term, investigated in the continuous case by Satsuma, and describing the coalescence of two travelling waves. We also derive the discrete form of a nondispersive evolution equation possessing like the Korteweg–de Vries equation a two-soliton solution.

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1. Introduction

The concept of a generalized conditional symmetry (GCS) was introduced and used by Fokas and Liu [1, 2] to obtain nonintegrable equations supporting multi-soliton and multi-kink solutions and particular examples are related to Burgers and Korteweg–de Vries (KdV) equations. For instance, the equation

$$u_t = \gamma u_{xx} + (1 - 3\gamma)uu_x + (1 - \gamma)(-u^3 + \alpha u + \beta) \tag{1.1}$$

which is nonintegrable for $\gamma \neq 1$ shares with the Burgers equation

$$u_t = K_1(u) \equiv u_{xx} - 2uu_x \tag{1.2}$$

the general solution of the nonlinear ordinary differential equation (ODE)

$$\sigma_1(u) \equiv u_{xx} - 3uu_x + u^3 - \alpha u - \beta = 0. \tag{1.3}$$

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Under the transformation $u = -\psi_x/\psi$, equation (1.2) is linearizable into

$$\psi_t - \psi_{xx} = 0 \tag{1.4}$$

while equation (1.3) is linearizable into

$$\psi_{xxx} - \alpha\psi_x + \beta\psi = 0 \tag{1.5}$$

with general solution

$$\psi = \sum_{i=1}^{3} A_i \exp(\lambda_i x) \qquad \lambda_i^3 - \alpha\lambda_i + \beta = 0 \quad A_i \text{ arbitrary.} \tag{1.6}$$

Therefore (1.3) represents the GCS for a two-kink solution. Now, equation (1.1) may be written as

$$u_t = K_1(u) + (\gamma - 1)\sigma_1(u) \tag{1.7}$$

and under the transformation $u = -\psi_x/\psi$ it can be associated with the linear equations

$$\psi_t - \psi_{xx} = 0 \qquad \psi_{xxx} - \alpha\psi_x + \beta\psi = 0 \qquad \gamma \neq 1. \tag{1.8}$$

Therefore

$$\psi = \sum_{i=1}^{3} \exp\left(\lambda_i x + \omega_i t + \delta_i\right) \qquad \omega_i = \lambda_i^2 \quad \delta_i \text{ arbitrary.} \tag{1.9}$$

Choosing $\beta = 0$, equation (1.1) possesses the particular solution

$$u = -\partial_x \mathrm{Log}\big(1 + \exp\theta_+ + \exp\theta_-\big) \qquad \theta_\pm = \pm\sqrt{\alpha}\, x + \alpha\, t + \delta_\pm \tag{1.10}$$

which describes the coalescence of two kinks depending on the fixed parameter α. This expression is also a solution of (1.2), with α being in this case an arbitrary constant.

For $\gamma = 1/3$ equation (1.1) is related to the Fitzhugh–Nagumo–Kolmogorov–Petrovskii–Piskunov (KPP) equation [3–6] which arises in population dynamics, nerve pulse propagation in nerve fibres and wall motion in liquid crystal while the case $1/3 < \gamma < 1$ was first investigated by Satsuma [7–9] who derived the particular solution (1.10), transforming the evolution equation (1.1) into a bilinear form of the variable ψ.

Two other interesting examples are the nonintegrable equations

$$u_t = -2uu_{xx} + 2u_x^2 - 2u^2u_x - 2\alpha u_x \tag{1.11}$$

and

$$u_t = u_x^2 + 2u^2u_x - 2\alpha u_x + u^4 + 2\alpha u^2 + 2\beta. \tag{1.12}$$

They share with the potential KdV equation

$$u_t = K_2(u) \equiv u_{xxx} + 6u_x^2 \tag{1.13}$$

the solution of the nonlinear ODE

$$\sigma_2(u) \equiv uu_{xx} - \frac{u_x^2}{2} + 2u^2u_x + \frac{u^4}{2} + \alpha u^2 + \beta = 0 \tag{1.14}$$

which represents the GCS for the two-soliton solution and is linearizable. Indeed, under the transformation $u = v_x/v$, equation (1.14) takes the bilinear form

$$v_x v_{xxx} - \tfrac{1}{2}v_{xx}^2 + \alpha v_x^2 + \beta v^2 = 0 \tag{1.15}$$

which by differentiation gives the fourth-order linear equation

$$v_{xxxx} + 2\alpha v_{xx} + 2\beta v = 0. \tag{1.16}$$

Therefore, equation (1.14) possesses the general solution

$$u = \partial_x \mathrm{Log}\left(\sum_{i=1}^{4} A_i \exp(\lambda_i x)\right) \qquad \lambda_i^4 + 2\alpha\lambda_i^2 + 2\beta = 0 \quad A_i \text{ arbitrary} \tag{1.17}$$

with $\lambda_1^2 A_1 A_2 = \lambda_3^2 A_3 A_4$. Equations (1.11) and (1.12) can be respectively written

$$u_t = K_2(u) - \frac{\partial_x \sigma_2(u)}{u} \tag{1.18}$$

$$u_t = K_2(u) + 2\sigma_2(u) - \frac{\partial_x \sigma_2(u)}{u} \tag{1.19}$$

and under the transformation $u = v_x/v$, they can be associated with the linear system

$$v_t + 2v_{xxx} + 6\alpha v_x = 0 \qquad v_{xxxx} + 2\alpha v_{xx} + 2\beta v = 0 \tag{1.20}$$

taking account of the first integral (1.15). Therefore, choosing $\alpha = -(\lambda_+^2 + \lambda_-^2)$ and $2\beta = (\lambda_+^2 - \lambda_-^2)^2$,

$$v = \sum_{\pm} A_{\pm} \exp\left((\lambda_+ \pm \lambda_-)x - 4(\lambda_+^3 \pm \lambda_-^3)t\right) + \sum_{\pm} B_{\pm} \exp\left(-(\lambda_+ \pm \lambda_-)x + 4(\lambda_+^3 \pm \lambda_-^3)t\right) \tag{1.21}$$

$$A_+ B_+ = A_- B_- \left(\frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-}\right)^2 \tag{1.22}$$

and one obtains for both equations (1.11) and (1.12) the particular solution

$$u = \lambda_+ + \lambda_- + \partial_x \mathrm{Log}(1 + \exp\theta_+ + \exp\theta_- + \kappa_{12}\exp(\theta_+ + \theta_-)) \tag{1.23}$$

$$\theta_{\pm} = -2\lambda_{\pm} x + 8\lambda_{\pm}^3 t + \delta_{\pm} \qquad \kappa_{12} = \left(\frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-}\right)^2. \tag{1.24}$$

This expression is also a solution of (1.13), with $\lambda_+ \neq \lambda_-$ being in this case some arbitrary constants.

The GCS (1.3) and (1.14) are particular cases of two linearizable nonlinear ODEs of the Gambier classification [10] usually referred to as G5 and G27. They may be obtained from the original definition of GCS which involves Fréchet differentiation, but can also be obtained using the Bäcklund transformation or the linearization of the related integrable partial differential equations.

In section 2 we use the latter technique to derive discrete GCS for the equations of the discrete Burgers hierarchy [11] in order to obtain the discrete analogue of the nonintegrable equation (1.1) which supports two-kink solutions. In section 3, we consider the so-called Lotka–Volterra equation having as the continuous limit the KdV equation and derive from its Bäcklund transformation the discrete equivalent of the GCS (1.14). This latter linearizable difference equation is used for building discrete equivalents of the nonintegrable equations (1.11) and (1.12) which both support a solution describing the elastic collision of two solitary waves with fixed speeds.

This paper is an updated version of [12].

2. Discrete Satsuma equation

The discretization of the Burgers hierarchy has been given by Levi *et al* [11] and may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}T} U(n) = \overline{K}(U) \equiv \sum_{j=1}^{M} \alpha_j [U(n+j) - U(n)] \prod_{l=0}^{j-1} U(n+l) \qquad M = 1, 2, 3, \ldots. \tag{2.1}$$

They may be linearized by the substitution,

$$U(n) = \phi(n+1)\phi(n)^{-1} \tag{2.2}$$

to give

$$\frac{\mathrm{d}}{\mathrm{d}T}\phi(n) = \sum_{j=1}^{M} \alpha_j \phi(n+j). \tag{2.3}$$

The N-kink solution corresponds to

$$\phi(n) = \sum_{l=1}^{N+1} A_l (\lambda_l)^n \exp\left[\sum_{j=1}^{M} \alpha_j (\lambda_l)^j T\right] \tag{2.4}$$

where $\{A_l\}$, $\{\lambda_l\}$ are sets of arbitrary real constants. For simplification, the T-dependence of U and ϕ is taken to be understood. It is straightforward to derive the nonlinear difference equation satisfied by the N-kink solution (corresponding to the GCS of Fokas and Liu [2] in the continuum case). Let $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ be the roots of the equation

$$\lambda^{N+1} + c_1\lambda^N + c_2\lambda^{N-1} + \cdots + c_N\lambda + c_{N+1} = 0 \tag{2.5}$$

for any given set $\{c_j\}$ of coefficients. Then

$$\phi(n+N+1) + c_1\phi(n+N) + \cdots + c_N\phi(n+1) + c_{N+1}\phi(n) = 0. \tag{2.6}$$

Dividing by $\phi(n)$,

$$\overline{\sigma}(U) \equiv \prod_{l=0}^{N} U(n+l) + c_1 \prod_{l=0}^{N-1} U(n+l) + \cdots + c_N U(n) + c_{N+1} = 0. \tag{2.7}$$

This is the nonlinear difference equation of order N satisfied by the N-kink solution of (2.1), given by (2.2) and (2.4). In analogy to the continuum case, this solution also satisfies the whole family of evolution equations

$$\frac{\mathrm{d}}{\mathrm{d}T}U(n) = \overline{K}(U) + G(U, \overline{\sigma}) \tag{2.8}$$

where $G(U, \overline{\sigma})$ is any function of $U(n)$ and $\overline{\sigma}(U)$ such that $G(U, 0) = 0$.

We now restrict ourselves to the case when $M = 2,\ \alpha_1 = 0,\ \alpha_2 = 1$ so that (2.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}T}U(n) = \overline{K}_2(U) \equiv [U(n+2) - U(n)]U(n+1)U(n) \tag{2.9}$$

which is a discrete form of the Burgers equation.

The two-kink solution of (2.9) is

$$U(n) = \frac{\phi(n+1)}{\phi(n)} \qquad \phi(n) = \sum_{l=1}^{3} A_l (\lambda_l)^n \exp(\lambda_l^2 T). \tag{2.10}$$

When the λ_l are the roots of

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \tag{2.11}$$

this solution satisfies the nonlinear difference equation

$$\overline{\sigma}_2(U) \equiv U(n)U(n+1)U(n+2) + c_1 U(n)U(n+1) + c_2 U(n) + c_3 = 0. \tag{2.12}$$

This equation represents a particular case of the discrete mapping obtained by Ramani *et al* [19] and Grammaticos *et al* [20] as a result of the discretization of two projective Riccati equations. Then, the equation

$$\frac{\mathrm{d}}{\mathrm{d}T}U(n) = [U(n+2) - U(n)]U(n+1)U(n) + G(U, \overline{\sigma}_2) \tag{2.13}$$

has the two-kink solution given by (2.10) when $G(U, 0) = 0$.

The discrete equations (2.9) and (2.12) have respectively as the continuous limit the Burgers equation (1.2) and the GCS (1.3). Indeed, making the replacement $n \to nh$, defining the change of variables

$$U(nh, T) = 1 - hu(x, t) \qquad x = nh + 2hT \quad t = 2h^2T \tag{2.14}$$

and letting $h \to 0$ in (2.9), with the product nh finite, the limit leads to

$$-2h^3(u_t - u_{xx} - 2uu_x) + \mathcal{O}(h^4) = 0. \tag{2.15}$$

On the other hand, setting in (2.12)

$$c_1 = -3 \qquad c_2 = 3 - \alpha h^2 \qquad c_3 = -1 + \alpha h^2 + \beta h^3 \tag{2.16}$$

the limit gives

$$\overline{\sigma}_2(U) \equiv -h^3[u_{xx} - 3uu_x + u^3 - \alpha u - \beta] + \mathcal{O}(h^4) = 0 \tag{2.17}$$

where the expression between the brackets is identical to (1.3).

The discrete analogue of (1.1)

$$\frac{\mathrm{d}}{\mathrm{d}T}U(n) = \overline{K}_2(U) + 2(\gamma - 1)\overline{\sigma}_2(U) \tag{2.18}$$

can explicitly be written, setting $h = 1$, as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}U(n) = {} & [U(n+2) - U(n)]U(n+1)U(n) + 2(\gamma - 1)[U(n)U(n+1)U(n+2) \\ & -3U(n)U(n+1) + (3-\alpha)U(n) + (\alpha + \beta - 1)].\end{aligned} \tag{2.19}$$

Under the transformation $U(n) = \phi(n+1)/\phi(n)$, the associated linear system is

$$\frac{\mathrm{d}}{\mathrm{d}T}\phi(n) = \phi(n+2) - \phi(n) \qquad \phi(n+3) + c_1\phi(n+2) + c_2\phi(n+1) + c_3\phi(n) = 0 \tag{2.20}$$

where c_1, c_2 and c_3 are given by (2.16). Therefore, choosing $\beta = 0$, a particular solution of (2.19) is

$$\frac{1 - U(nh)}{h} = -\sqrt{\alpha}\,\frac{\exp\theta(n, T, \sqrt{\alpha}) - \exp\theta(n, T, -\sqrt{\alpha})}{1 + \exp\theta(n, T, \sqrt{\alpha}) + \exp\theta(n, T, -\sqrt{\alpha})} \tag{2.21}$$

$$\theta(n, T, \sqrt{\alpha}) = n\mathrm{Log}(1 + \sqrt{\alpha}h) + 2\sqrt{\alpha}hT + \alpha h^2T. \tag{2.22}$$

Taking account of the relations in (2.14) the continuous limit for θ is

$$\lim_{h\to 0} \theta(n, T, \sqrt{\alpha}) = \sqrt{\alpha}(x + \sqrt{\alpha}\,t) \tag{2.23}$$

and the expression (2.21) tends to (1.10).

3. Discrete conditional symmetry for two solitons

Let us consider two discrete forms of the KdV equation. The first one is the so-called Lotka–Volterra equation, arising in prey–predator processes and in plasma physics [13, 14],

$$\frac{\mathrm{d}}{\mathrm{d}T}[\log N_n] = N_{n-1} - N_{n+1} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3.1}$$

which may be written in *potential form*,

$$\frac{\mathrm{d}}{\mathrm{d}T}[\log X_n] = 1 - \frac{X_{n+1}}{X_{n-1}} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3.2}$$

by taking

$$N_n(T) = \frac{X_{n+1}(T)}{X_{n-1}(T)} \tag{3.3}$$

and choosing the integration constant equal to 1. The integrability of (3.1) has been proven by Kac and van Moerbeke [16] and Manakov [17]. The second discrete equation was introduced by Hirota and Satsuma [15]

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{w_n}{1+w_n}\right) = w_{n-1/2} - w_{n+1/2} \tag{3.4}$$

which may in turn be written in the *potential form*

$$\frac{\mathrm{d}}{\mathrm{d}T}[\log Y_n] = \frac{Y_{n-1/2}}{Y_{n+1/2}} - 1 \qquad n = 0, \pm\tfrac{1}{2}, \pm 1, \ldots \tag{3.5}$$

by taking

$$w_n = \frac{Y_{n-1/2}}{Y_n} - 1 \tag{3.6}$$

and choosing the integration constant equal to -1. Equation (3.5) reduces to (3.2) by the transformation

$$n \rightarrow 2n \qquad Y_{2n} = X_n^{-1} \tag{3.7}$$

and related transformation [18]

$$N_n = (1 + w_{2n})(1 + w_{2n+1}). \tag{3.8}$$

Due to this equivalence, we only use equation (3.2) for deriving the discrete GCS. Making the transformation

$$X_n = f_{n+1/2}/f_{n-1/2} \tag{3.9}$$

equation (3.2) can be written in the bilinear form [21, 22]

$$[D_T \sinh(\tfrac{1}{2}D_n) + 2\sinh(D_n)\sinh(\tfrac{1}{2}D_n)]f_n \cdot f_n = 0 \tag{3.10}$$

with the definitions

$$D_T\, a \cdot b = a_T\, b - a\, b_T \qquad \exp(\epsilon D_n)\, a_n \cdot b_n = a(n+\epsilon)\, b(n-\epsilon) \tag{3.11}$$

and the corresponding Bäcklund transformation is

$$\exp(-\tfrac{1}{2}D_n) f_n \cdot \tilde{f}_n = [\lambda \exp(\tfrac{3}{2}D_n) + \mu \exp(\tfrac{1}{2}D_n)] f_n \cdot \tilde{f}_n \tag{3.12}$$

$$[D_T - \lambda \exp(2D_n) - \gamma] f_n \cdot \tilde{f}_n = 0 \tag{3.13}$$

where f_n and $\tilde{f}_n$ are two distinct solutions of (3.10).

Setting in the Bäcklund transformation $\lambda = \lambda_1$, $\mu = 1 - \lambda_1$, $\gamma = -\lambda_1$ and $f_n = 1$, the two linear equations

$$\tilde{f}_{n+1/2} = \lambda_1 \tilde{f}_{n-3/2} + (1 - \lambda_1)\tilde{f}_{n-1/2} \tag{3.14}$$

$$\tilde{f}_{n,T} + \lambda_1 \tilde{f}_{n-2} - \lambda_1 \tilde{f}_n = 0 \tag{3.15}$$

define the *one-soliton* solution which corresponds to the form

$$\tilde{f}_n = 1 + \exp\left(2(\omega_1 T - p_1 n) + \eta_1\right) \tag{3.16}$$

with $\lambda_1 = -\exp(-2p_1)$, $\omega_1 = \sinh 2p_1$ and p_1, η_1 arbitrary constants. The corresponding solution of (3.2) defined as $\tilde{X}_n = \tilde{f}_{n+1/2}/\tilde{f}_{n-1/2}$ satisfies the discrete Riccati equation

$$\overline{\sigma}_1(n) \equiv \tilde{X}_{n-1}\tilde{X}_n - \lambda_1 - (1 - \lambda_1)\tilde{X}_{n-1} = 0. \tag{3.17}$$

Interchanging the f_n and $\tilde{f}_n$ in (3.12) and setting $\mu = 1 - \lambda_2,\ \gamma = -\lambda_2$, we have

$$\tilde{f}_{n-1/2} f_{n+1/2} = \lambda_2 \tilde{f}_{n+3/2} f_{n-3/2} + (1 - \lambda_2) \tilde{f}_{n+1/2} f_{n-1/2} \tag{3.18}$$

where f_n now corresponds to the *two-soliton* solution.

Dividing through by $\tilde{f}_{n+1/2} f_{n-1/2}$, it becomes

$$\frac{X_n}{\tilde{X}_n} = \lambda_2 \frac{\tilde{X}_{n+1}}{X_{n-1}} + (1 - \lambda_2). \tag{3.19}$$

Eliminating $\tilde{X}_n$ between (3.17) for $\overline{\sigma}_1(n+1)$ and (3.19) gives

$$\tilde{X}_{n+1} = \frac{X_{n-1}[\lambda_1(1-\lambda_2) + (1-\lambda_1)X_n]}{[X_n X_{n-1} - \lambda_1\lambda_2]}. \tag{3.20}$$

Substituting back in (3.17) we obtain the following difference equation

$$\begin{aligned}\overline{\sigma}_2(n) \equiv{}& X_n\,(\lambda_1(1-\lambda_2) + (1-\lambda_1)X_{n+1})\,(\lambda_2(1-\lambda_1) + (1-\lambda_2)X_{n+1}) \\ &-(X_{n+1}X_n - \lambda_1\lambda_2)(X_{n+2}X_{n+1} - \lambda_1\lambda_2) = 0\end{aligned} \tag{3.21}$$

which corresponds to the discrete GCS for the two-soliton solution.

Now making the transformation (3.9) in (3.21), this equation becomes

$$\begin{aligned}E_n \equiv{}& (\lambda_1(1-\lambda_2) f_{n+1/2} + (1-\lambda_1) f_{n+3/2})(\lambda_2(1-\lambda_1) f_{n+1/2} + (1-\lambda_2) f_{n+3/2}) \\ &-(f_{n+3/2} - \lambda_1\lambda_2 f_{n-1/2})(f_{n+5/2} - \lambda_1\lambda_2 f_{n+1/2}) = 0.\end{aligned} \tag{3.22}$$

Therefore, the ratio

$$\begin{aligned}\frac{\lambda_1\lambda_2 E_{n-1} - E_n}{f_{n+3/2} - \lambda_1\lambda_2 f_{n-1/2}} \equiv{}& f_{n+5/2} - (1-\lambda_1)(1-\lambda_2) f_{n+3/2} \\ &-\lambda_1\lambda_2\left(\frac{(1-\lambda_2)^2}{\lambda_2} + \frac{(1-\lambda_1)^2}{\lambda_1} + 2\right) f_{n+1/2} \\ &-\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2) f_{n-1/2} + \lambda_1^2\lambda_2^2 f_{n-3/2} = 0\end{aligned} \tag{3.23}$$

is a linear difference equation of fourth order which corresponds to the linearization (1.16) in the continuum case. If one substitutes

$$x = nh \qquad \lambda_i = -1 + 2h\bar{\lambda}_i \qquad X_n = 1 + h\big(u(x) - (\bar{\lambda}_1 + \bar{\lambda}_2)\big) \tag{3.24}$$

in (3.21), it gives

$$-2h^4\left[u u_{xx} - \frac{1}{2}u_x^2 + 2u^2 u_x - (\bar{\lambda}_1^2 + \bar{\lambda}_2^2)u^2 + \frac{u^4}{2} + \frac{1}{2}(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2\right] + \mathcal{O}(h^5) = 0. \tag{3.25}$$

Dividing through by $-2h^4$ and letting $h \to 0$, this expression corresponds to the GCS (1.14) with $\alpha = -(\bar{\lambda}_1^2 + \bar{\lambda}_2^2),\ \beta = \frac{1}{2}(\bar{\lambda}_1^2 - \bar{\lambda}_2^2)^2$.

We now build two lattice equations which share with the integrable differential-difference equation (3.1) the two-soliton solution satisfying the GCS (3.21) and which possess as continuous limit respectively the nonintegrable partial differential equations (1.11) and (1.12).

First, we consider the continuous limit of equation (3.2). Taking

$$X_n(T) \equiv X(nh, T) = 1 + h(u(x,t) + c) \qquad x = nh - 2hT \quad t = -h^3T/3 \tag{3.26}$$

with c arbitrary, we obtain

$$-\tfrac{1}{3}h^4(u_t - u_{xxx} - 6u_x^2) + \mathcal{O}(h^5) = 0 \tag{3.27}$$

so that dividing through by $-h^4/3$ and letting $h \to 0$, we recover the potential KdV equation (1.13).

Then, we consider the lattice equation

$$\frac{\mathrm{d}}{\mathrm{d}T}X_n = X_n - \frac{X_{n+1}X_n}{X_{n-1}} - \frac{\overline{\sigma}_2(n-1)}{(X_nX_{n-1}-\lambda_1\lambda_2)X_{n-1}} \tag{3.28}$$

$$\equiv X_n - \frac{\lambda_1\lambda_2}{X_{n-1}} - \frac{(\lambda_1(1-\lambda_2)+(1-\lambda_1)X_n)\,(\lambda_2(1-\lambda_1)+(1-\lambda_2)X_n)}{(X_nX_{n-1}-\lambda_1\lambda_2)} \tag{3.29}$$

which is a discrete equation of first order possessing the two-soliton solution of (3.2). To get its continuous limit, instead of (3.28) we consider

$$\frac{\mathrm{d}}{\mathrm{d}T}X_n = X_n - \frac{X_{n+1}X_n}{X_{n-1}} - \frac{\frac{1}{3}(\overline{\sigma}_2(n)-\overline{\sigma}_2(n-1)) - \frac{h^2}{3}\overline{\sigma}_2(n) + \frac{4h^2}{3}\overline{\sigma}_2(n-1)}{(X_nX_{n-1}-\lambda_1\lambda_2)X_{n-1}} \tag{3.30}$$

which is identical to (3.28) for $h = 1$. Then, using the relations (3.26) with

$$c = -(\bar{\lambda}_1 + \bar{\lambda}_2) \qquad \lambda_i = -1 + 2h\bar{\lambda}_i \tag{3.31}$$

we have that

$$\frac{\overline{\sigma}_2(n)-\overline{\sigma}_2(n-1)}{(X_nX_{n-1}-\lambda_1\lambda_2)X_{n-1}} = -h^4\left\{u_{xxx} + 4u_x^2 + 2u_{xx}u - 2u_x(\bar{\lambda}_1^2+\bar{\lambda}_2^2) + 2u^2u_x\right\} + \mathcal{O}(h^5) \tag{3.32}$$

and the lattice equation (3.30) becomes

$$h^4\left\{u_t + 2u_{xx}u - 2u_x^2 + 2u^2u_x - 2(\bar{\lambda}_1^2+\bar{\lambda}_2^2)u_x\right\} + \mathcal{O}(h^5) = 0. \tag{3.33}$$

Dividing through by h^4 and letting $h \to 0$ we recover equation (1.11).

We now consider the expression

$$G(X,\overline{\sigma}_2) = \frac{\lambda_1\lambda_2\overline{\sigma}_2(n-1) - \overline{\sigma}_2(n)X_nX_{n-1}}{(X_{n+1}X_n-\lambda_1\lambda_2)\,X_{n-1}} + 2\frac{\overline{\sigma}_2(n-1)}{X_{n-1}} \tag{3.34}$$

$$\equiv \frac{(\lambda_1\lambda_2 - X_nX_{n-1})\overline{\sigma}_2(n-1)}{(X_nX_{n+1}-\lambda_1\lambda_2)X_{n-1}} - \frac{(\overline{\sigma}_2(n)-\overline{\sigma}_2(n-1))\,X_n}{X_nX_{n+1}-\lambda_1\lambda_2} + 2\frac{\overline{\sigma}_2(n-1)}{X_{n-1}} \tag{3.35}$$

which in the limit $h \to 0$ gives

$$G(X,\overline{\sigma}_2) \equiv -h^4(2\sigma_2(u) - \partial_x\sigma_2(u)/u) + \mathcal{O}(h^5) \tag{3.36}$$

where $\sigma_2(u)$ is the GCS (1.14). Therefore, the lattice equation

$$\frac{\mathrm{d}}{\mathrm{d}T}X_n = X_n - \frac{X_{n+1}X_n}{X_{n-1}} + \frac{1}{3}G(X,\overline{\sigma}_2) \tag{3.37}$$

becomes

$$h^4\left\{u_t - u_x^2 - 2u^2u_x - 2(\bar{\lambda}_1^2+\bar{\lambda}_2^2)u_x - u^4 + 2(\bar{\lambda}_1^2+\bar{\lambda}_2^2)u^2 - (\bar{\lambda}_1^2-\bar{\lambda}_2^2)^2\right\} + \mathcal{O}(h^5) = 0 \tag{3.38}$$

and dividing through by h^4, letting $h \to 0$, it corresponds to (1.12).

With transformation (3.9) and definition (3.22) for E_n one gets the identification

$$\overline{\sigma}_2(n) \equiv \frac{E_n}{f_{n+1/2}f_{n-1/2}} \tag{3.39}$$

$$G(X,\overline{\sigma}_2) \equiv \frac{\lambda_1\lambda_2E_{n-1} - E_n}{f_{n-1/2}(f_{n+3/2} - \lambda_1\lambda_2 f_{n-1/2})} + 2\,\frac{E_{n-1}}{f_{n-1/2}^2} \tag{3.40}$$

so that the lattice equation (3.37) can be written in the bilinear form

$$3\left(f_{n-1/2}\frac{\mathrm{d}}{\mathrm{d}T}f_{n+1/2} - f_{n+1/2}\frac{\mathrm{d}}{\mathrm{d}T}f_{n-1/2}\right) = (2\lambda_1\lambda_2 - 3)f_{n+3/2}f_{n-3/2}$$
$$+2(1-\lambda_1)(1-\lambda_2)f_{n+1/2}^2 - 2f_{n+3/2}f_{n+1/2} + f_{n+5/2}f_{n-1/2}$$
$$-(1-\lambda_1)(1-\lambda_2)f_{n+3/2}f_{n-1/2} + \left(3 + \lambda_1(1-\lambda_2)^2 + \lambda_2(1-\lambda_1)^2\right)$$
$$\times f_{n+1/2}f_{n-1/2} - \lambda_1^2\lambda_2^2 f_{n-3/2}f_{n-1/2} + \lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)f_{n-1/2}^2. \qquad (3.41)$$

This equation possesses the two-soliton discrete solution $f_n \equiv F^{(2)}(nh, T)$, with

$$F^{(2)}(nh, T) = 1 + \exp\Theta_1 + \exp\Theta_2 + \kappa_{12}(h)\exp(\Theta_1 + \Theta_2) \qquad (3.42)$$

$$\Theta_i(n, T) = -2p_i nh + 2\sinh(2hp_i)T \qquad (3.43)$$

$$\kappa_{12}(h) = \left(\frac{\lambda_1 - \lambda_2}{1 - \lambda_1\lambda_2}\right)^2 \qquad \lambda_i = -\exp(-2hp_i) \qquad i = 1, 2. \qquad (3.44)$$

In the continuous limit $h \to 0$, taking account of the change of variables (3.26), one recovers the two-soliton of the KdV equation

$$\lim_{h\to 0} F^{(2)}(nh, T) = 1 + \exp(\theta_1) + \exp(\theta_2) + \kappa_{12}(h)\exp(\theta_1 + \theta_2) \qquad (3.45)$$

$$\theta_i(x, t) = -2p_i x - 8p_i^3 t \qquad \kappa_{12} \equiv \lim_{h\to 0}\kappa_{12}(h) = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2. \qquad (3.46)$$

4. Conclusion

We have derived in this paper discrete equivalents of nonintegrable evolution equations which possess exact two-kink and two-soliton solutions. In the continuous case, although those equations take a bilinear form after a transformation on the dependent variable, they cannot be written only with Hirota differential operators. This feature is also valid for their discrete counterparts as is clear from (3.41) whose right-hand side cannot simply be written in terms of discrete Hirota operators.

It might be interesting to scrutinize the lattice equations to obtain other exact solutions and to extend this study to more than one-space dimension, both in continuous and discrete cases.

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